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U-Statistics of Random-Size Samples and Limit Theorems for Systems of Markovian Particles with Non-Poisson Initial Distributions

by

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Abstract

Limiting distributions of square-integrable infinite order U-statistics were first studied by Dynkin and Mandelbaum (1983) and Mandelbaum and Taqqu (1984). We extend their results to the case of non-Poisson random sample size. Multiple integrals of non-Gaussian generalized fields are constructed to identify the limiting distributions. An invariance principle is also established.

We use these results to study the limiting distribution of the amount of charge left in some set by an infinite system of signed Markovian particles when the initial particle density goes to infinity. By selecting the initial particle distribution, we determine the limiting distribution of charge, constructing different non-Gaussian generalized random fields, including Laplace, α-stable, and their multiple integrals.

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1 Introduction

This paper pursues two objectives: to study the asymptotic behaviour of symmetric statistics with random sample size, and to apply the resulting limit theorems for U-statistics to study the asymptotic behaviour of infinite particle systems with random non-Poisson initial distribution. Our main result is Theorem 1, which describes the asymptotic distribution of U-statistics in terms of multiple integrals of a non-Gaussian process, whose distribution is determined by the choice of the distribution of the sample size. The construction of these integrals is given in Sections 2 and 3 of the paper. One motivation for studying statistics with random size is that it is not always possible to take a fixed number of measurements. In queueing theory, reliability, and sequential analysis, study of statistics with random size goes back to the works of Rényi (1956), Robbins (1948a,b), Gnedenko and Fahim (1965) (see also the survey Gnedenko (1983) and the recent monographs Lee (1990), Kruglov and Korolev (1990) and Rachev (1991)). The rest of the introduction motivates the study of infinite particle systems with non-Poisson initial distribution. Thus, readers who are not interested in this second problem should skip directly to Section 2 of the paper.

In recent years much attention has been given to the description of infinite systems of particles moving according to some law (usually Markovian). Among these are works by Snitzman (1984), Shiga and Tanaka (1985), Walsh (1986), Adler and Epstein (1987), Adler (1989, 1990), Epstein (1989), Adler, Feldman and Lewin (1991), and others.

Many of these papers deal with particle systems which behave as follows: Initially (at time zero) a number of independent particles pop into existence at locations within the space R^d, according to a Poisson point process with intensity λ. The particles then move about according to some Markov law. The asymptotic behaviour of this system as $\lambda \to \infty$ has been studied in Martin-Löf (1976), Itô (1983), Walsh (1986, Ch.8), Adler and Epstein (1987), Adler (1989, 1990), Adler, Feldman and Lewin (1991) for different conditions. In particular, Adler and Epstein (1987) obtain convergence of sums of some functionals of the Markov processes to generalized Gaussian random fields and their functionals. The authors show how these limit theorems can be used to study properties of the limiting random fields.

The question we ask in this paper is, "What happens to a Markovian particle system if we change the initial distribution of the particles?" When the Central Limit Theorem is applied to a sum of N i.i.d. random variables, non-Poisson randomization of the sample size N leads to non-Gaussian limits (see, for example, Rachev (1991), Section 19). The choice of a non-Poisson initial distribution, e.g., geometric or "discretized" α-stable, produces non-Gaussian generalized random fields as limits of sums of some functionals of Markov processes, and this construction provides a tool for the study

of these fields.

Note that the limiting distributions which we obtain have many practical applications. Laplace processes, which can be generated via a geometric summation scheme, are used in reliability (Brown (1990), Gertsbakh (1990)), in enviromental studies (Rachev and Todorovich (1991)), and in modeling of financial data (Mittnik and Rachev (1990)). The recent developments and applications of stable processes are covered in Samorodnitsky and Taqqu (1990), Rachev and Rüschendorf (1990), etc. Blattberg and Genodes (1974) observed that the t-distribution provides a better model for "peaky" distributions than the Gaussian does. Melamed (1989) studies the generalized Laplace distribution. By selecting the initial distribution of the particles, we are able to produce Laplace, stable, generalized Laplace, "t", and other generalized random fields. Note that one-dimensional time processes of these types were obtained in Mandelbrot and Taylor (1967) and Clark (1973) and used for modeling stock returns, providing better fits than Gaussian processes do.

We now describe several interesting ways in which particles may be born on R^d, $d \geq 1$. Let R^d be divided into unit cubes with vertices on the lattice Z^d. On a probability space take a Poisson random variable $N(\lambda)$ with mean $\lambda > 0$. At the initial time, $N(\lambda)$ particles are born independently in each cube, and are distributed uniformly. Thus, at the initial time we observe what we will call a "Poisson picture" in R^d. Alternatively, the particles could be generated according to the following scheme: Imagine a generator which at each step remains active with probability q. If active, it produces one particle (uniformly distributed) in each cube. The particles are held in the cubes where they were born until the time of a "catastrophe", a geometrically distributed moment, when the generator fails. At that time particles become free to move over the whole space R^d according to some Markov law until their exponential lifetimes expire. Note that the starting time $t = 0$ of the system is the moment when the generator fails. We are interested in the case where the failure probability $1 - q$ is very small, in which case the average density of particles is very large.

The geometric and Poisson distributions both have the property that the probability of a large number x of particles being born in one cube approaches zero at an exponential rate as $x \to \infty$. It might be interesting to consider a system in which the initial particle density can take very large values with high probability. This requires a distribution with a heavy tail. As an example, we use a "discretized" version of a positive stable distribution. Another interesting initial distribution is a mixture of Poisson distributions. This means that the particles are generated by several Poisson distributions, one of which might produce most of the particles. We also consider mixed empirical and doubly stochastic point process of particles, each having finite initial measure.

Following the construction of Adler and Epstein (1987), we shall assign a Rademacher

A-1

positive or negative charge to each particle, send an appropriate parameter of the initial distribution to infinity and study the limiting distribution of charge left by the system in a set after all particles have died. The limiting field, which is indexed by sets, or more generally, by functions, will have a non-Gaussian distribution. As special cases, we obtain Laplace, stable, Gamma, "t", and other fields. We also construct their functionals using limits of sums of symmetric functionals of the Markov processes in the system.

This paper is organized as follows: First, we develop some general limit theorems for sums of symmetric functionals of independent random variables with values in an arbitrary measure space when the number of summands is random. For fixed and Poisson sample size, such theorems were proved by Dynkin and Mandelbaum (1983) and an invariance principle was established by Mandelbaum and Taqqu (1984) in their studies of U-statistics. We describe our generalizations of their results in Section 2; proofs are given in Section 3. In Sections 4 and 5 we analyze the distribution of charge left in the space by systems of Markov processes created under different initial conditions.

2 Symmetric statistics with random sample size and multiple integrals.

This section contains our results on symmetric statistics, often called U-statistics, when the number of summands is random. Let $X, X_1, X_2, \ldots$ be i.i.d. random variables taking values in an arbitrary measurable space $(\mathcal{X}, \mathcal{F})$ with distribution ν. For each $T > 0$, let $\mathcal{H}_T$ be a space of sequences of functions $\{h_k; k \geq 0\}$ for which h_0 is a constant, $h_k = h_k(x_1, \ldots, x_k)$ is a symmetric function on $\mathcal{X}^k$ such that

$$||h||_T^2 := \sum_{k=0}^{\infty} \frac{T^k}{k!} \nu^k(h_k^2) < \infty,$$

where

$$\nu^k(h_k^2) := \int \int_{\mathcal{X}^k} h_k^2(x_1, \ldots, x_k) \nu(dx_1) \ldots \nu(dx_k).$$

Let $\mathcal{H} = \cap_{T>0} \mathcal{H}_T$. The function h_k will be called canonical if

$$\int_{\mathcal{X}} h_k(x_1, \ldots, x_{k-1}, x) \nu(dx) = 0 \quad \nu^{k-1} - a.e.$$

Let $\{h_k, k \geq 0\}$ be a sequence of canonical functions from $\mathcal{H}$. Define sums

$$\sigma_k^n(h_k) := \sum \cdots \sum_{1 \leq i_1 < \ldots < i_k \leq n} h_k(X_{i_1}, \ldots, X_{i_k}) \tag{1}$$

for $n \geq k$ and $\sigma_k^n(h_k) := 0$ otherwise.

The limiting distribution of $n^{-k/2}\sigma_k^n$ as $n \to \infty$ was studied by Dynkin and Mandelbaum (1983) (as well as by other authors). They also considered statistics (1) for Poisson sample size, i.e. the limit as $\lambda \to \infty$ of $\lambda^{-k/2}\sigma_k^{N_\lambda}$, where N_λ is a Poisson random variable with mean $\lambda > 0$ independent of $X_1, X_2, \ldots$. In both cases Dynkin and Mandelbaum obtain the same limits, which are written in terms of multiple Wiener integrals of Gaussian measure defined on $(\mathcal{X}, \mathcal{F})$.

For $r > 0$, let $N(r)$ be a positive random variable taking integer values independently of the X_i's and such that

$$N(r)/r \xrightarrow{\mathcal{D}} Y \text{ as } r \to \infty \tag{2}$$

for some positive random variable Y independent of X's. We would like to study the limiting behavior of statistics $r^{-k/2}\sigma_k^{N(r)}$ as $r \to \infty$. The limiting distribution in this case appears to be expressed via multiple integrals with respect to a random measure $\{M(B), B \in \mathcal{F}, \nu(B) < \infty\}$ which has orthogonal increments.

We will denote by $\stackrel{\mathcal{D}}{\Rightarrow}$ weak convergence of finite dimensional distributions.

The main result of this section is the following theorem:

Theorem 1. *Let $h = \{h_k, k \geq 0\}$ be a sequence of canonical functions in $\mathcal{H}$. Let $\mathcal{L}_Y$ be the Laplace transform of Y. As $r \to \infty$ the finite dimensional distributions of*

$$Z_r(h) := \sum_{k=0}^{\infty} r^{-k/2}\sigma_k^{N(r)}(h_k) \tag{3}$$

converge to those of $\sum_{k=0}^{\infty}(1/k!)J_k(h_k)$, where J_k are multiple integrals with respect to a random measure M on $(\mathcal{X}, \mathcal{F})$ such that for sets $B_1, \ldots, B_n \in \mathcal{F}$ the vector $(M(B_1), \ldots, M(B_n))$ has the characteristic functional

$$Ee^{i\lambda_1 M(B_1)+\ldots+i\lambda_n M(B_n)} = \mathcal{L}_Y\left(\frac{1}{2}\sum_{i,j=1}^{n} \lambda_i\lambda_j\nu(B_i \cap B_j)\right). \tag{4}$$

The construction of the multiple integrals J_k will be given later, in the course of the proof. We will use a technique which has parallel in stochastic analysis, where many results on continuous local martingales can be obtained via quadratic variation time change from results on Brownian motion (see Revuz and Yor (1991)); our main tool will be a "random function change" in the Wiener integrals with respect to Gaussian measure, defined in Dynkin and Mandelbaum (1983).

Before we proceed into technicalities, let us give some examples of random measures M which correspond to different distributions of sample size $N(r)$.

Example 1. (Poisson sample size). If $N(r)$ is a Poisson random variable with mean $r > 0$, then $N(r)/r \to 1$ and M coincides with the Gaussian measure $\{W(B), B \in \mathcal{F}, \nu(B) < \infty\}$ such that $EW(B) = 0, EW(A)W(B) = \nu(A \cap B)$. This is the case considered by Dynkin and Mandelbaum (1983) and Mandelbaum and Taqqu (1984).

Example 2. (Geometric sample size). Let $N(r)$ be a geometric random variable with mean r. Then Y is a standard exponential and M is a Laplace random measure $\{L(B), B \in \mathcal{F}, \nu(B) < \infty\}$ such that

$$Ee^{i\lambda_1 L(B_1)+\ldots+i\lambda_n L(B_n)} = \frac{1}{1+\frac{1}{2}\sum_{i,j=1}^{n} \lambda_i\lambda_j \nu(B_i \cap B_j)}. \tag{5}$$

Example 3. (Mixture of Poisson). Define $N^j(r)$, $j = 1, 2, \ldots, k$, as a sequence of independent Poisson r.v's with means $r\alpha_j$, $\alpha_j > 0, j = 1, 2, \ldots, k$. Consider a mixture of Poisson distributions

$$P(N(r) = l) = \sum_{j=1}^{k} p_j P(N^j(r) = l),$$

where $p_j \geq 0, \sum_{j=1}^{k} p_j = 1$. In this case Y is a discrete random variable with $P(Y = \alpha_j) = p_j$ and the measure M is a mixture of Wiener measures (Kon (1984) applies *such distributions to model stock returns)*.

Example 4. (Discretized stable sample size). Let Y be a positive stable r.v. with Laplace transform $Ee^{-\lambda Y} = e^{-\lambda^{\alpha/2}}, \lambda > 0$, where the index of Y $\alpha/2$ is less than 1. Let $N(r)$ be the following discretized version of Y:

$$P(N(r) = k) = P(k-1 < rY \leq k),\ k \geq 1 \tag{6}$$

Clearly, (2) holds and the random measure M is symmetric stable with parameter α:

$$Ee^{i\lambda_1 M(B_1)+\ldots+i\lambda_n M(B_n)} = exp\left\{-\left(\frac{1}{2}\sum_{i,j=1}^{n} \lambda_i\lambda_j \nu(B_i \cap B_j)\right)^{\alpha/2}\right\}.$$

Note, that the measure M is different from the stable random measures studied in Weron (1984) and Samorodnitsky and Taqqu (1990), since its increments are not independent.

Example 5. Let $1/Y$ be a chi-square r.v. and let $N(r)$ be a discretized version (6) of Y. For $B \in \mathcal{F}$, $M(B)$ has a t-distribution.

Example 6. (Generalized geometric sample size). Pick any $m \geq 0$. Let $N(r)$ have generalized geometric distribution (cf. Melamed (1989))

$$P(N(r) = 1 + km) = \frac{1}{k!}\prod_{j=0}^{k-1}(\frac{1}{r} + j)(\frac{1}{r})^{1/m}(1 - \frac{1}{r})^k \ \text{for } k \geq 1$$

and $P(N(r) = 1) \sim r^{-1/m}$. Then Y in (2) is a Gamma$(1/m, m)$ distributed r.v. with Laplace transform $Ee^{-\lambda Y} = (1 + m\lambda)^{-1/m}$. M has the distribution

$$Ee^{i\lambda_1 M(B_1)+\ldots+i\lambda_n M(B_n)} = \left(1 + \frac{m}{2}\sum_{i,j=1}^{n} \lambda_i\lambda_j \nu(B_i \cap B_j)\right)^{m/2}.$$

We now rigorously construct and characterize the random measure M and its multiple integrals. For the Gaussian case this was done by Dynkin and Mandelbaum (1983) and extended to an invariance principle by Mandelbaum and Taqqu (1984). We proceed in steps, in order to show how the defined integrals appear naturally in the study of the limiting distribution of $Z_r(h)$ of Theorem 1.

On some probability space, define a linear random family $\{J_1(\phi), \phi \in L^2(\nu)\}$ through its finite-dimensional distributions given by

$$E\,exp\{i\lambda J_1(\phi)\} = \mathcal{L}_Y\left(\frac{1}{2}\lambda^2\nu(\phi^2)\right), \tag{7}$$

and

$$aJ_1(\phi) + bJ_1(\psi) = J_1(a\phi + b\psi) \text{ a.s. for all } a, b \in R \tag{8}$$

where $\mathcal{L}_Y$, as before, is the Laplace transform of the r.v. Y. Note that although $J_1 = J_1^Y$ depends on Y, we will suppress the index Y in our notation. The family J_1 is defined on a probability space, which may be different from the one which supports the X_i's and Y, but we will use the same sign for expectations. If we consider the subfamily

$$\{M(B) = J_1(1_B),\ B \in \mathcal{F},\ \nu(B) < \infty\},$$

then symbolically

$$J_1(\phi) = \int_{\mathcal{X}} \phi(x)M(dx).$$

Lemma 1 below shows the relationship between the statistics $\sigma_1^{N(r)}$ and the family J_1.

Let $L = \{\phi \in L^2(\nu):\ \nu(\phi) = 0\}$.

Lemma 1. *Under the conditions of Theorem 1,*

$$\left\{r^{-1/2}\sum_{i=1}^{N(r)} \phi(X_i)\right\}_{\phi\in L} \stackrel{\mathcal{D}}{\Rightarrow} \{J_1(\phi)\}_{\phi\in L} \tag{9}$$

Our next step is to define an analog of quadratic variation of the process J_1. As is seen from Lemma 2 below, this role will be played by the linear random family

$\{K(\psi), \psi \in L^2(\nu)\}$, which is defined on the same probability space as the family $J_1(\phi)$, and whose joint distribution with $J_1(\phi)$ is given by

$$E\, exp\{i\lambda J_1(\phi) + i\mu K(\psi)\} = \mathcal{L}_Y\left(-i\mu\nu(\psi^2) + \frac{1}{2}\lambda^2\nu(\phi^2)\right). \tag{10}$$

Here the Laplace transform $\mathcal{L}_Y$ is defined on the right half-plane ($\mathrm{Re}\ z \geq 0$).

Lemma 2. *As $r \to \infty$,*

$$\left\{\left(r^{-1/2}\sum_{i=1}^{N(r)}\phi(X_i),\ r^{-1}\sum_{i=1}^{N(r)}\psi^2(X_i)\right)\right\}_{\phi,\psi\in L} \stackrel{\mathcal{D}}{\Rightarrow} \{(J_1(\phi), K(\psi))\}_{\phi,\psi\in L}. \tag{11}$$

As the next step, for $\phi \in L$ define $h_0^\phi = 1, \ldots, h_k^\phi(x_1, \ldots, x_k) = \phi(x_1)\cdot \ldots \cdot \phi(x_k), \ldots$. Then the sequence $h^\phi = \{h_k^\phi, k \geq 0\}$ is in the space $\mathcal{H}$ ($\mathcal{H}$ was defined at the beginning of this section) and $Z_r(h^\phi)$ is well defined. We are now ready to find its limiting distribution. Set

$$\varepsilon(\phi) = exp\{J_1(\phi) - \frac{1}{2}K(\phi)\} \tag{12}$$

Lemma 3. *As $r \to \infty$*

$$\left\{Z_r(h^\phi)\right\}_{\phi\in L} \stackrel{\mathcal{D}}{\Rightarrow} \{\varepsilon(\phi)\}_{\phi\in L} \tag{13}$$

Recall that the defined function ε is a generating function of generalized Hermite polynomials which serve as multiple integrals (cf., Revuz and Yor (1990)). Specifically, let $H_k(x)$ be the Hermite polynomial of order k with leading coefficient 1 :

$$H_k(x) = e^{x^2/2}(-1)^k\frac{d^k}{dx^k}\left(e^{-x^2/2}\right).$$

For $a > 0$, set

$$H_k(x,a) = a^{k/2}H_k(x/\sqrt{a}).$$

We also set $H_k(x,0) = x^k$. Then,

$$\sum_{k=0}^{\infty}\frac{u^k}{k!}H_k(x,a) = exp\{ux - au^2/2\}.$$

Recall that for a real-valued local martingale $\{A_t, t \geq 0\}$, $A_0 = 0$, with quadratic variation $< A, A >$, the iterated stochastic integral is defined via a generalized Hermite polynomial:

$$k!\int_0^t dA_{s_1}\int_0^{s_1} dA_{s_2}\ldots\int_0^{s_{k-1}} dA_{s_k} := H_k(A_t, < A, A >_t)$$

(Revuz and Yor, p.143). Similarly, define the multiple integral of order k on the space of functions $\{h_k^\phi, \phi \in L\}$ as

$$J_k(h_k^\phi) := H_k(J_1(\phi), K(\phi)) \tag{14}$$

Using the last definition, linearity of the processes J_1 and K, (12), and Lemma 3 we obtain

Lemma 4. *Under the conditions of Theorem 1,*

$$\left\{Z_r(h^\phi)\right\}_{\phi\in L} \stackrel{\mathcal{D}}{\Rightarrow} \left\{\sum_{k=0}^{\infty} \frac{1}{k!} J_k(h_k^\phi)\right\}_{\phi\in L} \tag{15}$$

as $r \to \infty$.

Since the space $\{h_k^\phi, k \geq 0, \phi \in L\}$ is dense in $\mathcal{H}$, the passage from Lemma 4 to Theorem 1 is possible.

The proofs of Theorem 1 and Lemmas 1,2, and 3 will follow in Section 3.

We will extend the result of Theorem 1 to an invariance principle for random sums of symmetric statistics, similar to one obtained by Mandelbaum and Taqqu (1984). Let $f : R_+ \to R$; $< f, f >_2 := \int_0^\infty f^2(x)dx < \infty$, $\phi \in L$. We extend the definition of the random integral to the product space of functions $L \times L^2(R_+)$ by putting

$$E\,exp\,\{i\lambda_1 J_1^{\nu\times Leb}(\phi_1 f_1) + \ldots + i\lambda_k J_1^{\nu\times Leb}(\phi_k f_k)\} = \tag{16}$$
$$\mathcal{L}_Y\left(\frac{1}{2}\sum_{i,j=1}^{k} \lambda_i\lambda_j \nu(\phi_i\phi_j) < f_i, f_j >_2\right).$$

$$(\phi_j, f_j) \in L \times L^2(R_+).$$

Then, the random measure M on the product space $(\mathcal{X}, \mathcal{F}, \nu) \times (R_+, \mathcal{B}(R_+), Leb)$ ($\mathcal{B}(R_+)$ is the Borel σ-field on positive half line) can be defined as follows:

$$M(B \times [0,s]) = J_1^{\nu\times Leb}(1_B 1_{[0,s]}),\ \ B \in \mathcal{F}, s > 0.$$

On $\mathcal{X}^k \times R_+^k$ define

$$h_{k,t}(x_1, \ldots, x_k, u_1, \ldots, u_k) := h_k(x_1, \ldots, x_k) 1_{[0,t]}(u_1) \ldots 1_{[0,t]}(u_k). \tag{17}$$

We then have:

Theorem 2. *Let* $\{h_k, k \geq 0\}$ *be a sequence of canonical functions in* $\mathcal{H}$. *As* $r \to \infty$, *the process*

$$Z_r^t(h) := \sum_{k=0}^{\infty} r^{-k/2} \sigma_k^{[N(r)t]}(h_k),\ t \geq 0, \tag{18}$$

converges weakly in $D[0,\infty)$ to

$$M^t(h) := \sum_{k=0}^{\infty} \frac{1}{k!} J_k^{\nu \times Leb}(h_{k,t}), \tag{19}$$

where the multiple integrals are taken with respect to the random measure M on the product space $(\mathcal{X}, \mathcal{F}, \nu) \times (R_+, \mathcal{B}(R_+), Leb)$ defined above.

Symbolically,

$$J_k^{\nu \times Leb}(h_{k,t}) = \int h_k(x_1, \ldots, x_k) 1_{[0,t]}(u_1) \ldots 1_{[0,t]}(u_k) M(dx_1, du_1) \ldots M(dx_k, du_k).$$

where the integral is taken over the product $(\mathcal{X} \times R_+)^k$.

3 Proofs of Theorems 1 and 2.

Let $\phi_1, \phi_2, \ldots, \psi_1, \psi_2, \ldots$ be functions in L.

Proof of Lemma 2. Consider the 2k-dimensional process

$$(\xi_r(t), \eta_r(t)) := \sum_{i=1}^{[rt]} (r^{-1/2} \underline{\phi}_i, r^{-1} \underline{\psi}_i^2),$$

$$\underline{\phi}_i := (\phi_1(X_i), \ldots, \phi_k(X_i)), \; \underline{\psi}_i^2 := (\psi_1^2(X_i), \ldots, \psi_k^2(X_i))$$

on $D[0, \infty)$.

From Donsker's invariance principle, the law of large numbers and Theorem 4.4 on p. 27 of Billingsley (1968) follows the weak convergence of (ξ_r, η_r), as $r \to \infty$, to a 2k-dimensional process $G_{(\phi,\psi)}$ such that

$$E\, exp\left\{i(\underline{\lambda}, \underline{\mu}) G_{(\phi,\psi)}(t)\right\} = exp\left\{it \sum_{i=1}^{k} \mu_i \nu(\psi_i^2) - \frac{t}{2} \sum_{i,j=1}^{k} \lambda_i \lambda_j \nu(\phi_i \phi_j)\right\}$$

$$\underline{\lambda} = (\lambda_1, \ldots, \lambda_k), \; \underline{\mu} = (\mu_1, \ldots, \mu_k)$$

From (2) and the independence of $N(r)$ and the X_i's, it follows that $(N(r)/r, (\xi_r, \eta_r))$ converges weakly to $(Y, G_{(\phi,\psi)})$.

By the Skorohod-Dudley theorem (cf., Dudley (1989), Theorem 11.7.2), there exists a probability space rich enough to support random pairs $\left(\overline{N}(r)/r, (\overline{\xi}_r, \overline{\eta}_r)\right)$ and

$\left(\overline{Y}, \overline{G}_{(\phi,\psi)}\right)$ having the same distribution as $(N(r)/r, (\xi_r, \eta_r))$ and $\left(Y, G_{(\phi,\psi)}\right)$, respectively, and such that

$$d\left(\left(\overline{\xi}_r(\frac{\overline{N}(r)}{r}), \overline{\eta}_r(\frac{\overline{N}(r)}{r})\right), \overline{G}_{(\phi,\psi)}(\overline{Y})\right) \to 0 \text{ a.s.}$$

where d is the Skorohod metric in $D[0,\infty)$ (see Resnick (1987), p. 221, Rachev and Rüschendorf (1990)).

Then,

$$\left(\xi_r(\frac{N(r)}{r}), \eta_r(\frac{N(r)}{r})\right) \xrightarrow{w} G_{(\phi,\psi)}(Y)$$

This immediately gives the result of Lemma 2.

Proof of Lemma 1. Lemma 1 follows immediately from Lemma 2 when $\psi \equiv 0$.

Proof of Lemma 3. For any $\phi \in L$,

$$\begin{aligned} log\, Z_r(h^\phi) &= log\left(\prod_{i=1}^{N(r)}(1 + r^{-1/2}\phi(X_i))\right) \\ &= \frac{1}{\sqrt{r}}\sum_{i=1}^{N(r)}\phi(X_i) - \frac{1}{2r}\sum_{i=1}^{N(r)}\phi^2(X_i) + o_\phi. \end{aligned}$$

($o_\phi \xrightarrow{\mathcal{D}} 0$ as $r \to \infty$.) In other words the field $\left\{log\, Z_r(h^\phi)\right\}_{\phi\in L}$ has the same limiting distribution (in the sense of weak convergence of finite dimensional distributions) as the field

$$\left\{\frac{1}{\sqrt{r}}\sum_{i=1}^{N(r)}\phi(X_i) - \frac{1}{2r}\sum_{i=1}^{N(r)}\phi^2(X_i)\right\}_{\phi\in L}.$$

Applying Lemma 2 to the latter field we obtain its convergence to $\left\{J_1(\phi) - \frac{1}{2}K(\phi)\right\}_{\phi\in L}$. This proves Lemma 3.

Lemma 4 was obtained in Section 2. We shall now prove Theorem 2; Theorem 1 will follow from Theorem 2 when $t = 1$.

Proof of Theorem 2. On an arbitrary measurable space $(\mathcal{X}', \mathcal{F}', \nu')$, define a Gaussian family $\{I_1(\phi), \phi \in L^2(\mathcal{X}')\}$ with

$$EI_1(\phi) = 0;\ EI_1(\phi)I_1(\psi) = \nu'(\phi\psi).$$

The multiple Wiener integral of order k associated with the Gaussian family I_1 is defined as a linear mapping I_k from the space $\mathcal{H}^k$ of symmetric functions $h_k(x_1, \ldots, x_k)$, $(\nu')^k(h_k^2) < \infty$, into the space of functionals of the Gaussian family I_1. The mapping is uniquely defined by the following conditions (cf., Dynkin and Mandelbaum (1983)):

A. $I_k(h_k^\phi) = H_k(I_1, \nu'(\phi^2))$, $\phi \in L^2(\mathcal{X}')$.

B. For $h_k \in \mathcal{H}^k$, $EI_k^2(h_k) = k!(\nu')^k(h_k^2)$.

For $\{h_k, k \geq 0\} \in \mathcal{H}$ define the multiple Wiener integral of the function $h_{k,t}$ ($h_{k,t}$ was defined in (17)). The integral is defined on the product space $(\mathcal{X}', \mathcal{F}', \nu') = (\mathcal{X}, \mathcal{F}, \nu) \times (R_+, \mathcal{B}(R_+), Leb)$ and is associated with the Gaussian family $\{I_1^{\nu \times Leb}(\phi f), (\phi, f) \in L^2(\mathcal{X}) \times L^2(R_+)\}$, which has mean zero and variance $\nu(\phi^2) < f, f >_2$ (cf., (16)).

Then, for $\{h_k, k \geq 0\} \in \mathcal{H}$, as $r \to \infty$

$$V_r^t(h) := \sum_{k=0}^{\infty} r^{-k/2} \sigma_k^{[rt]}(h_k),\ t \geq 0$$

converges weakly in $D[0, \infty)$ to

$$W^t(h) := \sum_{k=0}^{\infty} \frac{1}{k!} I_k^{\nu \times Leb}(h_{k,t}),\ t \geq 0$$

(Mandelbaum and Taqqu (1984)).

As in the proof of Lemma 2, the Skorohod-Dudley theorem and the result on p. 221 of Resnick (1987) yield

$$\left\{ V_r^{\frac{N(r)}{r}t}(h) \right\}_{t \geq 0} \xrightarrow{w} \left\{ W^{Yt}(h) \right\}_{t \geq 0},\ \text{as } r \to \infty$$

To complete the proof of Theorem 2 we have to show that

$$W^{Y\cdot}(h) \stackrel{\mathcal{D}}{=} M^{\cdot}(h) \tag{20}$$

From (10) and the independence of Y and I_1 it follows that $J_1(\phi)$ is equal in distribution to $\sqrt{Y} I_1(\phi) \stackrel{\mathcal{D}}{=} I_1(\sqrt{Y}\phi)$ (the latter is defined on the product of probability spaces), and $K(\phi) \stackrel{\mathcal{D}}{=} Y\nu(\phi^2) \stackrel{\mathcal{D}}{=} \nu\left((\sqrt{Y}\phi)^2\right)$. In particular, for the random measure M we have

$$M(B \times [0, s]) \stackrel{\mathcal{D}}{=} J_1^{\nu \times Leb}(1_B 1_{[0,s]}) \stackrel{\mathcal{D}}{=} \sqrt{Y} I_1^{\nu \times Leb}(1_B 1_{[0,s]}) \stackrel{\mathcal{D}}{=} I_1^{\nu \times Leb}(1_B 1_{[0,sY]}). \tag{21}$$

The last relationship follows from the facts that Y is independent from $I_1^{\nu \times Leb}$ and that a Gaussian distribution is fully determined by its mean and covariance.

We complete the construction of the multiple integral of order k with respect to the measure M on an arbitrary measurable space as follows. The comparison of the

generating functions for J_k and I_k for $h^\phi, \phi \in L$, yields

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{u^k}{k!} J_k(h_k^\phi) &= exp\{uJ_1(\phi) - \frac{u^2}{2} K(\phi)\} \\ &= exp\{u\sqrt{Y} I_1(\phi) - \frac{u^2}{2} Y \nu(\phi^2)\} \\ &= \sum_{k=0}^{\infty} \frac{u^k}{k!} Y^{k/2} I_k(h_k^\phi).\end{aligned}$$

Thus, we define $J_k(h_k) \stackrel{\mathcal{D}}{=} Y^{k/2} I_k(h_k)$ for $h_k \in \mathcal{H}^k$.

Recalling relationship (21) on the product space $(\mathcal{X}, \mathcal{F}, \nu) \times (R_+, \mathcal{B}(R_+), Leb)$ and the definitions of $W^Y(h)$ and $M(h)$, we conclude that (20) holds. This completes the proof of Theorem 2.

Proof of Theorem 1. Take $t = 1$ in the statement of Theorem 2. Due to the Cramér-Wold device (Billingsley (1968), p. 49) and the linearity of $Z_\tau^1(h)$ and $M^1(h)$ in the argument h, Theorem 1 follows if we prove that $J_k^{\nu \times Leb}(h_{k,1}) \stackrel{\mathcal{D}}{=} J_k(h_k)$. However, this follows immediately from the definition of J_k via I_k (see proof of Theorem 2) and the relationship $I_1(\phi) \stackrel{\mathcal{D}}{=} I_1^{\nu \times Leb}(\phi 1_{[0,1]})$. (Both variables are Gaussian with mean zero and variance $\nu(\phi^2)$.) This completes the proof of Theorem 1.

4 The limit theorems and random fields.

We now return to the particle picture described in the Introduction. Let $Z_d := \{n : n = (n_1, \ldots, n_d)\}$, i.e. the set of all d-dimensional integer-valued multi-indices. For each $n \in Z_d$ let C_n be the d-cube defined by $C_n = \{x \in R^d : n_i - 1 \leq x_i < n_i, i = 1, \ldots, d\}$. Let $p_t(x,y) = p_t(y,x), x, y \in R^d, t \geq 0$ be a symmetric Markov transition density function satisfying $\int_{R^d} p_t(x,y)dy = 1$ for each $x \in R^d$. Let g be the corresponding Green's function

$$g(x,y) = \int_0^\infty e^{-t} p_t(x,y)dt. \tag{22}$$

On a probability space $(\Omega, \mathcal{F}, P)$, define an infinite collection $\hat{X} = \{X_n(t), t \geq 0\}_{n \in Z_d}$ of independent symmetric Markov processes on R^d with common transition density $p_t(x,y)$, each process starting according to a uniform distribution in C_n. Furthermore, let the probability space be rich enough to support an infinite sequence $\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_i, \ldots$ of such collections, all independent of each other. If X is one of the processes in the entire collection (when both $n \in Z_d$ and $i > 0$ vary), then we can think about it as describing the movement of a particle in the system.

We take (on the same probability space) a sequence $\hat{\sigma}_1, \hat{\sigma}_2, \ldots$ of collections $\hat{\sigma}_i$ $\{\sigma_{i,n}, n \in Z_d\}$ of independent Rademacher variables, i.e. $P(\sigma_{i,n} = 1) = P(\sigma_{i,n} = -1) = 1/2$. We can think about $\sigma_{i,n}$ as a positive or negative "charge" associated with the Markov particle $X_{i,n}$. For additional motivation of this choice of "positive" and "negative" particles we refer to Adler (1989).

Extend the probability space to support a random variable $N(r)$, $r > 0$, independent of the X's and σ's and such that (2) holds. $N(r)$ represents the number of collections $\hat{X}_1, \ldots, \hat{X}_{N(r)}$ in the system at the initial time $t = 0$. We now describe the evolution of the system in time. When a particle with charge σ at time t passes through a point x in the space R^d, it leaves there a charge $e^{-t}\sigma$. Let $A \in \mathcal{B}(R^d)$ be a Borel set in the space R^d. We are interested in finding the amount of charge left in A after all particles have lost their charge and in the limit of increasing initial particle density, i.e. we would like to find a limiting distribution of

$$\Phi_r(A) := \frac{1}{\sqrt{r}} \sum_{i=1}^{N(r)} \sum_n \int_0^\infty \sigma_{i,n} e^{-t} 1_A(X_{i,n}(t)) dt \tag{23}$$

as $r \to \infty$. More generally, consider a bilinear form

$$< f, h > \equiv < f, h >_g := \int\int_{R^{2d}} f(x) g(x,y) h(y) dx dy,$$

where g is given by (22). Define the class of functions

$$S_d \equiv S_d(g) := \{ f : f \text{ on } R^d \text{ with } < |f|, |f| > < \infty \}.$$

We study the weak convergence, as $r \to \infty$, of the finite dimensional distributions of the sum

$$\Phi_r(f) := \frac{1}{\sqrt{r}} \sum_{i=1}^{N(r)} \sum_n \int_0^\infty \sigma_{i,n} e^{-t} f(X_{i,n}(t)) dt \tag{24}$$

Define on some probability space a generalized random field $\{\Phi(f), f \in \mathcal{S}_d\}$. This means (cf. Walsh (1986), p. 332):

(a) $\Phi(af + bh) = a\Phi(f) + b\Phi(h)$ a.s. for all $f, h \in S_d$, i.e. Φ is a linear random functional;

(b) Φ has a version with values in the dual space S_d'.

Corollary 4.2 of Walsh (1986), shows that in order to assure (b), Φ has to be continuous in probability. We specify the distribution of Φ via

$$E\, exp\{i\lambda\Phi(f)\} = \mathcal{L}_Y\left(\frac{1}{2}\lambda^2 < f, f >\right). \tag{25}$$

Lemma 5. *A linear random functional Φ with distribution (25) is continuous in probability on S_d.*

The proof of Lemma 5 is postponed to the end of the section.

Theorem 3. *As $r \to \infty$, the finite dimensional distributions of the field $\{\Phi_r(f), f \in S_d\}$ converge weakly to those of the field $\{\Phi(f), f \in S_d\}$.*

We will give the proof of Theorem 3 later.

The limit theorems developed in Section 2 allow us to build multiple integrals of the field $\Phi(f)$. To see this, consider k Markov processes $X_1, \ldots, X_k$ from the system. Define

$$F_{f_k}(X_1, \ldots, X_k) := \int_0^\infty \ldots \int_0^\infty e^{-t_1 - \ldots - t_k} f_k(X_1(t_1), \ldots, X_k(t_k)) dt_1 \ldots dt_k \tag{26}$$

for each function f_k from the space

$$S_d^k \equiv S_d^k(g) := \{f_k : f_k \text{ on } R^{dk} \text{ with } < |f_k|, |f_k| >< \infty\},$$

where

$$< f_k, h_k >:= \int\int_{R^{2dk}} f_k(x_1, \ldots, x_k) g(x_1, y_1) \ldots g(x_k, y_k) h(y_1, \ldots, y_k) dx_1 \ldots dx_k dy_1 \ldots dy_k.$$

We study the limiting distribution, as $r \to \infty$, of the sum

$$\Psi_r(f_k) := r^{-k/2} \sum \ldots \sum\nolimits_{1 \le i_1 < \ldots < i_k \le N(r)} \hat{F}_{f_k}(\hat{X}_{i_1}, \ldots, \hat{X}_{i_k}) \tag{27}$$

$$\hat{F}_{f_k}(\hat{X}_{i_1}, \ldots, \hat{X}_{i_k}) := \sum_{n_1} \ldots \sum_{n_k} \sigma_{i_1, n_1} \ldots \sigma_{i_k, n_k} F_{f_k}(X_{i_1, n_1}, \ldots, X_{i_k, n_k}) \tag{28}$$

Theorem 4. *As $r \to \infty$, the finite dimensional distributions of the pair $< \Phi_r(f), \Psi_r(f_k) >$ on $S_d \times S_d^k$ converge weakly to the finite dimensional distributions of the pair $< \Phi(f), (1/k!)\Psi(f_k) >$, where $\Psi(f_k)$ is the multiple integral of order k associated with the field $J_1(f) \stackrel{\mathcal{D}}{=} \Phi(f)$.*

Proofs of Theorems 3 and 4 are based on Theorem 1 and the following lemma:

Lemma 6. *Let $f_k, h_k \in S_d^k$. The functional $\hat{F}_{f_k}$ of (28) is square-integrable and*

$$E\hat{F}_{f_k}\hat{F}_{h_k} = < f_k, h_k > .$$

Proof. For simplicity, take $k = 1$. For $f, h \in S_d$

$$E\hat{F}_f\hat{F}_h = E \sum_n \sigma_n \int_0^\infty e^{-t} f(X_n(t)) dt \sum_m \sigma_m \int_0^\infty e^{-s} h(X_m(s)) ds \tag{29}$$

Since σ_n, σ_m are independent and with zero mean $E\,\sigma_n \sigma_m = \delta_{n,m}$ and (29) is equal to

$$E \sum_n \int_0^\infty \int_0^\infty e^{-t-s} f(X_n(t)) h(X_n(s)) dtds = \tag{30}$$
$$2 \int_0^\infty \int_t^\infty e^{-t-s} dtds \sum_n \int_{C_n} da \int_{R^d} p_t(a,x) f(x) dx \int_{R^d} p_{s-t}(x,y) h(y) dy.$$

Note that $p_t(a,x) = p_t(x,a)$,

$$\sum_n \int_{C_n} p_t(x,a) dx = 1$$

and that

$$2 \int_0^\infty \int_t^\infty e^{-t-s} p_{s-t}(x,y) dtds = g(x,y).$$

Thus, (30) is equal to

$$\int \int_{R^{2d}} f(x) g(x,y) h(y) dxdy \equiv < f, h > .$$

The proof for general $k > 1$ follows from similar arguments involving longer formulas but no new mathematics and we feel free to omit it. Lemma 6 is proved.

Proof of Theorem 3. Let the space $\mathcal{X} = \left((R^d)^{R_+} \times \{-1,1\}\right)^{Z_d}$ be the path space of pairs $(\hat{X}, \hat{\sigma})$, and denote by ν the probability measure the above pair induces on $\mathcal{X}$. Then by Lemma 1 the finite dimensional distributions of the sum (24) converge weakly to those of the field $J_1(\hat{F}_f)$, which is determined by

$$E\,exp\{i\lambda J_1(\hat{F}_f)\} = \mathcal{L}_Y\left(\frac{1}{2}\lambda^2 \nu(\hat{F}_f^2)\right).$$

However, $\nu(\hat{F}_f^2) \equiv E\hat{F}_f^2 =< f, f >$ by Lemma 6. Thus, $\{J_1(\hat{F}_f), f \in S_d\} \stackrel{\mathcal{D}}{=} \{\Phi(f), f \in S_d\}$. This completes the proof of Theorem 3.

Proof of Theorem 4. Let $\mathcal{X}$ and ν be as in the proof of Theorem 3. Theorem 4 follows immediately from Theorem 1, noticing that $\{J_1(\hat{F}_f), f \in S_d\} \stackrel{\mathcal{D}}{=} \{\Phi(f), f \in S_d\}$ and so that also $J_k \stackrel{\mathcal{D}}{=} \Psi$.

It follows from the above theorems and Examples 1–6 of Section 2 that:

(i) If $N(r)$ is a Poisson r.v. with mean r, the random field $\{\Phi(f), f \in S_d\}$ is a centered Gaussian with covariance $E\,\Phi(f)\Phi(h) =< f, g >$. (This is the case studied by Adler and Epstein (1987).)

(ii) If $N(r)$ is a geometric r.v. with mean r, the random field $\{\Phi(f), f \in S_d\}$ has the Laplace distribution

$$E exp\{i\lambda\Phi(f)\} = \frac{1}{1 + \frac{1}{2}\lambda^2 < f, f >}.$$

(iii) If $N(r)$ is a discretized stable r.v., as defined in (6), the random field $\{\Phi(f), f \in S_d\}$ is a generalized stable random field with distribution given by

$$Ee^{i\lambda\Phi(f)} = exp\left\{-\left(\frac{1}{2}\lambda^2 < f, f >\right)^{\alpha/2}\right\}.$$

We similarly obtain random fields whose marginal distributions include the t-distribution, Gamma distribution, and others.

Our last proof is of Lemma 5:

Proof of Lemma 5. Because Φ is linear, it is enough to prove that

$$\Phi(f_n) \xrightarrow{\mathcal{P}} 0 \text{ as } < f_n, f_n > \to 0.$$

Take $\epsilon > 0$. For any $\delta > 0$ take N large enough so that $P(Y > N) < \delta/2$ and n large enough so that $< f_n, f_n > < \delta\epsilon^2/2N$. Then from the relationship

$$\Phi(f_n) \stackrel{\mathcal{D}}{=} J_1(\hat{F}_f) \stackrel{\mathcal{D}}{=} \sqrt{Y} I_1(\hat{F}_f),$$

$(I_1(\hat{F}_f)$ is a centered generalized Gaussian random field indexed by $f \in S_d$ with variance $E(I_1(\hat{F}_f))^2 =< f, f >)$ and from the Chebychev inequality it follows that $P(|\Phi(f_n)| > \epsilon) < \delta$. The Lemma is proved.

5 Other initial distributions.

The results of Section 2 can be interpreted via point process terminology (cp., Dynkin and Mandelbaum (1983), p. 742): a sample $X_1, ..., X_{N(r)}$ can be viewed as a mixed empirical point process defined, for example, in Karr (1991), p. 7. Of course, if N(r) is Poisson, we have a Poisson point process. Theorem 1 then says that some functionals represented as multiple integrals with respect to the random measure M (given by (4)) are approximated by the functional (3) of the mixed empirical process. Similarly, Theorems 3 and 4 state that the generalized random field Φ and its multiple integrals can be approximated by functionals of the point process on the path space of cadlag functions, which is constructed in Section 4. If instead we consider a mixed empirical point process on the path space we get the following result:

Proposition 1. *Let X_i, $i = 1, 2, ...,$ be Markov processes as in Section 4, but with initial probability measure μ. Let $N(r)$ be as in Section 4. Define the following functionals of the processes:*

$$\Phi_r(f) = \frac{1}{\sqrt{r}} \sum_{i=1}^{N(r)} \int_0^\infty \sigma_i e^{-t} f(X_i(t))dt \tag{31}$$

$$\Psi_r(f_k) := r^{-k/2} \sum \cdots \sum_{1 \le i_1 < \cdots < i_k \le N(r)} \sigma_{i_1} \cdots \sigma_{i_k} F_{f_k}(X_{i_1}, \ldots, X_{i_k}) \tag{32}$$

where the functional F_{f_k} was defined in (26). Let $\{\Phi(f), f \in S_d\}$ be the generalized random field whose distribution is specified via (25) with $< f, f >$ replaced by

$$< f, f >_\mu := 2 \int_{R^{3d}} \mu(da)dxdy f(x)f(y)g^{(2)}(a,x)g(x,y); \tag{33}$$

$$g^{(2)}(x,y) := \int_0^\infty e^{-2t} p_t(x,y)dt.$$

Then, the statement of Theorem 4 holds.

Proof. As in the proofs of Theorems 3 and 4, we apply Theorem 1 to a sample living on the product of the path space of the Markov processes and $\{-1, 1\}$. We only have to establish that EF_f^2 is given by formula (33).

$$\begin{aligned} E \int_0^\infty \int_0^\infty e^{-t-s} f(X(t))f(X(s))dtds = \\ 2 \int_0^\infty \int_t^\infty e^{-t-s}dtds \int_{R^d} \mu(da) \int_{R^d} p_t(a,x)f(x)dx \int_{R^d} p_{s-t}(x,y)f(y)dy. \end{aligned} \tag{34}$$

Making the change of variables $t = u, s - t = v$ gives

$$2 \int_0^\infty \int_0^\infty e^{-2u-v}dudv \int_{R^d} \mu(da) \int_{R^d} p_u(a,x)f(x)dx \int_{R^d} p_v(x,y)f(y)dy.$$

Using Fubini's theorem and the expressions for g and $g^{(2)}$ given above, we obtain the right hand side of (33). This finishes the proof of Proposition 1.

Remark: Following Martin-Löf (1976) we can construct a stationary mixed empirical point process of Markovian particles with invariant measure $r \times$ Lebesgue. In the notation of Section 4, this would correspond to the case where the numbers of particles in each cube C_n are i.i.d. random variables, each distributed as $N(r)$. (In the construction of Section 4, $N(r)$ has the same value for all cubes.) We conjecture that a limit theorem similar to Proposition 1 will also hold in this case; however, this remains to be proven.

As a special case of Proposition 1, we obtain the limit for functionals of the processes which constitute a Cox point process in the path space.

Corollary. *Let $X_i, i = 1, 2, \ldots$, be as in Proposition 1. Let $\Lambda(r)$ be a positive random variable independent of the X_i's such that*

$$\frac{\Lambda(r)}{r} \xrightarrow{\mathcal{D}} Y \text{ as } r \to \infty, \tag{35}$$

for some positive random variable Y.
Let P_Λ be a random variable such that conditioned on Λ, P_Λ is Poisson with mean Λ. Then

$$\Phi_r(f) := \frac{1}{\sqrt{r}} \sum_{i=1}^{P_\Lambda} \int_0^\infty \sigma_i e^{-t} f(X_i(t))dt \overset{\mathcal{D}}{\Rightarrow} \Phi(f)$$

$$\Psi_r(f_k) := r^{-k/2} \sum \cdots \sum_{1 \le i_1 < \ldots < i_k \le P_\Lambda} \sigma_{i_1} \ldots \sigma_{i_k} F_{f_k}(X_{i_1}, \ldots, X_{i_k}) \overset{\mathcal{D}}{\Rightarrow} \Psi(f_k),$$

where $\{\Phi(f), f \in S_d\}$ is the generalized random field with distribution given in Proposition 1 and $\{\Psi(f_k), f_k \in S_d^k\}$ is the multiple integral of order k associated with Φ.

Proof: Note that as $r \to \infty$,

$$\frac{P_\Lambda}{r} = \frac{P_\Lambda}{\Lambda(r)} \frac{\Lambda(r)}{r} \overset{\mathcal{D}}{\to} Y$$

Thus, application of Proposition 1 with $N(r) = P_\Lambda$ completes the proof.

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